A birational description of the minimal exponent

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Setup: X smooth irred alg var$/\mathbb{C}$, dim n

$\emptyset \neq Z = (f=0) \hookrightarrow X$ hypersurface

Log resolution of (X,Z):

$$\begin{array}{ccc} & & Y \text{ smooth} \\ & & \downarrow \pi \\ Z & \hookrightarrow & X \end{array}$$

π proper

isom over $X \setminus Z$

$\pi^*(Z) = \sum a_i E_i$ SNC $\quad K_{Y/X} = \sum k_i E_i$, $E = \sum E_i$

Typical invar. of sing in birat geom:

$\mathrm{lct}(X,Z) := \sup\{c>0 \mid (X, cZ) \text{ klt}\}$

$= \min_i \frac{k_i+1}{a_i}$

$= \sup\{c>0 \mid \frac{1}{|f|^{2c}} \text{ is locally integrable}\}$

$= \min\{c>0 \mid \mathcal{J}(X,cZ) \neq \mathcal{O}_X\}$

where $\mathcal{J}(X,cZ) = \pi_* \mathcal{O}_Y(K_{Y/X} - \lfloor c\cdot \pi^*(Z)\rfloor)$ mult. ideal

Goal: get a similar description for the <u>minimal exponent</u>

We will see: instead just of $K_{Y/X}$, this will involve also

$\Omega_Y^p(\log E)$ for $p<n$

§1. The definition of the minimal exponent

<u>Thm</u> (Bernstein, Kashiwara) $\exists\, b(s) \neq 0$ s.t.

$$b(s) f^s \in \mathcal{D}_X[s] \cdot f^{s+1} \qquad \text{where } \mathcal{D}_X = \text{sheaf of diff. ops on } X$$

$$\text{and} \quad \partial_{x_i} \cdot f^s = s\, \frac{\partial f / \partial x_i}{f} \cdot f^s \quad \forall i$$

<u>Example</u> Z smooth $\Rightarrow$ have local coord $f = x_1, x_2, \dots x_n$

$$\partial_{x_1} \cdot f^{s+1} = (s+1) f^s$$

The <u>b-function</u> $b_f(s)$ is the monic generator of the ideal of $b(s)$ as in thm.

Kashiwara, Lichtin: given a log res as before, every root of $b_f(s)$ is of the form $-\dfrac{k_i + \ell}{a_i}$ for some i, $\ell \in \mathbb{Z}_{>0}$

$$\Rightarrow \ \leq -\operatorname{lct}(X, Z)$$

Analyt. descr of lct + integr. by parts $\Rightarrow$ this is in fact equality (Kollár)

<u>Rmk</u>: If we make $s = -1$ in

$$b_f(s) f^s = P(s) \cdot f^{s+1} \Rightarrow b_f(-1) \frac{1}{f} \in \mathcal{O}_X$$

Hence $b_f(-1) = 0$

<u>Def</u> (Saito) The minimal exp. $\tilde{\alpha}(Z)$ is $-$(largest root of $b_f(s)/(s+1)$)

$$\in \mathbb{Q}_{>0} \cup \{\infty\}$$

$$\Rightarrow \quad \operatorname{lct}(X, Z) = \min\{\tilde{\alpha}(Z), 1\}$$

Facts:
$$\begin{cases} \tilde{\alpha}(Z) = \infty & \text{iff } Z \text{ smooth} \\ \tilde{\alpha}(Z) > 1 & \text{iff } Z \text{ has rational singularities} \end{cases}$$

- $\tilde{\alpha}(Z)$: refinement of lct(X,Z) interesting when Z has rational sing.
- controls "higher order versions" of $\begin{smallmatrix}\text{Du Bois}\\ \text{rational}\end{smallmatrix}$ singularities

Suppose now we have a log resol $\pi : Y \longrightarrow X$ of (X,Z) as before

<u>Thm 1</u> If $p \in \mathbb{Z}_{>0}$, then $\tilde{\alpha}(Z) > p$ iff

1) $R^q \pi_* \Omega_Y^i(\log E) = 0 \quad \forall q \geq 1, \; i \leq p$

2) $\operatorname{codim}_Z(Z_{sing}) \geq 2p$

(enough: $\geq \min\{3, 2p\}$)

This is a variant of

$\tilde{\alpha}(Z) > p \iff Z$ has p-rational sing

M-Popa Saito (def. of Friedman-Laza involving a resolution of Z)

<u>Thm 2</u> If $p \in \mathbb{Z}_{>0}$ is s.t $\tilde{\alpha}(Z) \geq p$ and $\alpha \in (0,1)$, then

- $R^q \pi_* \Omega_Y^{n-p}(\log E)(-E - \lfloor \alpha \pi^*(Z) \rfloor)$
  $\rightarrow R^q \pi_* \Omega_Y^{n-p}(\log E)(-E) \quad (*)$
  is isom $\forall q \neq p$, inj for $q = p$
- if $\tilde{\alpha}(Z) > p$, then $(*)$ is bij for $q = p$ iff $\tilde{\alpha}(Z) > p + \alpha$

Rmk: If $p = 0 \Rightarrow$ the above condition says
$\operatorname{lct}(X, Z) > \alpha$ iff
$$\omega_X \otimes \mathcal{J}(X, \alpha Z) \hookrightarrow \omega_X \quad \text{isom}$$

Rmk: Using Groth. duality, the cond in Thm2
$\Leftrightarrow$ if $\tilde{\alpha}(Z) > p$, then $> p + \alpha$ iff

$$R^q \pi_* \Omega_Y^p(\log E)(L_\alpha D_1) = 0 \quad \forall q \geq 1$$

§2. The minimal exponent and the V-filtration

$$\text{If} \quad i : X \hookrightarrow X \times \mathbb{A}^1_t, \quad x \longmapsto (x, f(x))$$

$$B_f := i_+ \mathcal{O}_X = \mathcal{H}^1_{i(X)} \mathcal{O}_{X \times \mathbb{A}^1}$$

$$= \mathcal{O}_X[t]_{f-t} / \mathcal{O}_X[t]$$

$$= \bigoplus_{j \geq 1} \mathcal{O}_X \cdot \frac{1}{(f-t)^j}$$

This is a module / $\mathcal{D}_X \langle t, \partial_t \rangle$

It carries 2 filtrations:

- the Hodge filtration:

$$F_p B_f = \bigoplus_{j \leq p} \mathcal{O}_X \cdot \frac{1}{(f-t)^j}$$

- the V-filtration of Malgrange, Kashiwara:

$(V^\alpha B_f)_{\alpha \in \mathbb{Q}}$ decreasing, exhaustive by D_X-mod

const. on $\left(\frac{i}{\ell}, \frac{i+1}{\ell}\right]$, $i \in \mathbb{Z}$

(for some ℓ)

uniquely char by some properties, the most important being

$$\partial_t t - \alpha \quad \text{is nilpotent on } Gr_V^\alpha B_f = V^\alpha / V^{>\alpha}$$

- construction uses existence of b_f + rationality of its roots:

$$\mathcal{O}_X[s, \tfrac{1}{f}] f^s \quad \simeq \quad i_+ \mathcal{O}_X[\tfrac{1}{f}]$$

$$\cup$$

$$i_+ \mathcal{O}_X$$

$$f^s \longleftrightarrow \frac{1}{f-t}$$

s-action $\longleftrightarrow$ action of $-\partial_t t$

Why it is important:

$$\psi_f(\mathcal{O}_X) = \bigoplus_{\alpha \in (0,1]} \underbrace{Gr_V^\alpha B_f}_{\psi_{f,\alpha}(\mathcal{O}_X)}$$

D-module theoretic nearby cycles

Thm (Saito) If $p \in \mathbb{Z}_{\geq 0}$, $\alpha \in (0,1] \cap \mathbb{Q}$

$\Rightarrow$ $\tilde{\alpha}(Z) \geq p + \alpha$

iff $F_{p+1} B_f \subseteq V^{\alpha} B_f$

Recall the def of $Gr_p^F DR_X(\mathcal{M})$ when

$(\mathcal{M}, F)$ $\mathcal{D}_X$-module with good filtration:

$$DR_X(\mathcal{M}): \quad \overset{-n}{0} \to \mathcal{M} \xrightarrow{\nabla} \Omega_X^1 \otimes \mathcal{M} \xrightarrow{\nabla} \cdots \cdots \xrightarrow{\nabla} \Omega_X^n \otimes \mathcal{M}$$

linear / $\mathbb{C}$

$$Gr_{p-n}^F DR_X(\mathcal{M}): \quad 0 \to Gr_{p-n}^F \mathcal{M} \to \Omega_X^1 \otimes Gr_{p-n}^F \mathcal{M} \to \cdots$$

linear / $\mathcal{O}_X$ $\in D^b_{coh}(X)$

Saito's thm $\Rightarrow$ if $\tilde{\alpha}(Z) \geq p$, $\alpha \in (0,1)$

- $Gr_i^F \psi_{f,\alpha}(\mathcal{O}_X) = 0$ $\forall\ i \leq p$
- If $\tilde{\alpha}(Z) \geq p + \alpha$, then $\tilde{\alpha}(Z) > p + \alpha$

iff $Gr_{p+1}^F \psi_{f,\alpha}(\mathcal{O}_X) = 0$

iff $Gr_{p+1-n}^F DR_X \psi_{f,\alpha}(\mathcal{O}_X) = 0$

(all $\mathcal{H}^q$ with $q \neq 0$ are autom 0)

Suppose now $\begin{array}{ccc} Y & \hookleftarrow & E \\ \pi\downarrow & & \downarrow \\ X & \hookleftarrow & Z \end{array}$ is a log res. as in our setup

$g = f\circ\pi$

Basic fact: $\quad \psi_{f,\alpha}(\mathcal{O}_X) = \pi_+ \psi_{g,\alpha}(\mathcal{O}_Y) \quad \forall\, \alpha > 0$

$\Rightarrow Gr_p^F DR_X \psi_{f,\alpha}(\mathcal{O}_X) \simeq R\pi_* Gr_p^F DR_Y \psi_{g,\alpha}(\mathcal{O}_Y) \quad \forall\, p$

$\Rightarrow$ for our criterion, we need to compute

$$Gr_p^F DR_Y \psi_{g,\alpha}(\mathcal{O}_Y)$$

This is doable since $\mathrm{div}(g)$ is SNC. This is our main technical result:

define $\Omega_{Y/\mathbb{A}^1}(\log E)$ via

$$0 \to \mathcal{O}_Y \xrightarrow{d\log(g)} \Omega_Y^1(\log E) \to \Omega^1_{Y/\mathbb{A}^1}(\log E) \to 0$$

$D_\alpha = \lceil \alpha\, \pi^*(Z) \rceil \Rightarrow$ a filtered resol. of

$((V^\alpha B_g)^r, F_\cdot[-1])$ is given by

$0 \to \mathcal{O}_Y(-D_\alpha)\otimes \mathcal{D}_Y \to \mathcal{O}_Y(-D_\alpha)\otimes \Omega^1_{Y/\mathbb{A}^1}(\log E)\otimes \mathcal{D}_Y \to \cdots$

$\cdots \to \mathcal{O}_Y(-D_\alpha)\otimes \Omega^{n-1}_{Y/\mathbb{A}^1}(\log E)\otimes \mathcal{D}_\alpha \to 0$

$\rightsquigarrow$ get similar filtered res. for $\psi_{g,\alpha}(\mathcal{O}_Y)^r$

Using the fact that $Gr^F_{i-n} DR_Y(\mathcal{D}_Y) \simeq \begin{cases} \omega_Y & \text{if } i=0 \\ 0 & \text{otherwise} \end{cases}$

we get $\quad Gr^F_{i-n+1} DR_X \psi_{f,\alpha}(\mathcal{O}_X)$

$$\simeq$$

$$R\pi_* \left(\mathcal{O}_Y(-D_\alpha)/\mathcal{O}_Y(-D_{>\alpha})\right)\otimes_{\mathcal{O}_Y} \Omega^{n-1-i}_{Y/\mathbb{A}^1}(\log E)[i]$$

The pf. of Thm 2 follows using this + prev. criterion for $\tilde{\alpha}(Z) > p+\alpha$